DERIVATION OF FORMULAS FOR THE GRADIENT OF THE ERROR IN THE ITERATIVE SOLUTION OF INVERSE PROBLEMS OF HEAT CONDUCTION. I. DETERMINATION OF THE GRADIENT IN TERMS OF THE GREEN'S FUNCTION

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We consider the construction of the gradient of the error functional for the iterative solution of inverse problems involving equations of the parabolic type. The linear formulation of the problem is analyzed.

In the solution of incorrectly posed inverse problems of mathematical physics, a widely used method is iterative regularization, where the stability of the approximate solution is attained by limiting the number of iterations in a way that is consistent with errors in the initial data [1-6]. Regularizing gradient algorithms [2, 3, 6-11] possess high computational efficiency and universality of practical application. In this method, the gradient of the error functional is calculated in each iteration

$$J(u) = \frac{1}{2}\left\| Au - f \right\|_F^2, \qquad (1)$$

where this functional is associated with the extremum formulation of the inverse problem

$$Au = f,\ u \in U,\ f \in F.$$

Here u and f are the unknown and known elements: U and F are certain normalized spaces; $A:U \to F$ is a given operator. The procedure of calculating the gradient J'_u determines to a significant degree the accuracy and speed of the solution algorithm of the inverse problem.

The present paper is one of three parts. We consider two general analytical approaches to the construction of the error gradient for inverse problems of generalized heat conduction. The first is based on the integral representation of the solution in terms of a Green's function and is applicable in the case of a linear operator A (this approach is given in the present paper). The second approach applies not only to linear problems, but also to nonlinear ones, and is based on the solution of a boundary-value problem conjugate to the problem for the field increment of the variable state (this approach will be considered in the second and third papers of this series). These methods of finding the gradient are distinguished by their high accuracy and fast execution time.

Below we consider the usual case in practical situations where the space of square-integrable L_2 functions is chosen as U and F. We consider only the case of a single spatial variable. As shown in [12], it is a rather simple matter to transform from a known gradient of the functional in the space L_2 to the gradient of this functional in the space of W_2^k functions, which, along with their derivatives up to the k-th order (the derivative is understood in the sense of generalized functions) are square-integrable. Then a priori information on the smoothness of the required function can be taken into account

The determination of the error gradient in terms of the Green's function assumes that the solution of the corresponding direct problem is known, and written in terms of this function. We discuss this matter in more detail.

In the region $Q = (0, b) \times (0, \tau_m)$ we consider the following parabolic equation with variable coefficients

$$L_{x\tau} T(x, \tau) = q(x, \tau), \qquad (2)$$

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where

$$L_{x\tau}=\frac{\partial}{\partial\tau}-A_{x\tau},\ A_{x\tau}=a_1(x,\ \tau)\frac{\partial^2}{\partial x^2}+a_2(x,\ \tau)\frac{\partial}{\partial x}+a_3(x,\ \tau).$$

We consider (2) subject to the initial condition

$$T(x,\ 0)=\lim_{\tau\to 0+}T(x,\ \tau)=\xi(x) \tag{3}$$

and the boundary conditions

$$B_{1\tau}T(0,\ \tau)=p_1(\tau),\ B_{2\tau}T(b,\ \tau)=p_2(\tau), \tag{4}$$

where the operators $B_{1\tau}$ and $B_{2\tau}$ have the form:

$$B_{1\tau}=\left[\gamma_1(\tau)\frac{\partial}{\partial x}+\sigma_1(\tau)\right]_{x=0}, \tag{5}$$

$$B_{2\tau}=\left[\gamma_2(\tau)\frac{\partial}{\partial x}+\sigma_2(\tau)\right]_{x=b}. \tag{6}$$

We assume that $\gamma_1 \neq 0$, $\gamma_2 \neq 0$. The corresponding results for boundary conditions of the first kind can be obtained in a similar way and will be presented without detailed derivation.

We consider the system of operators $\{A_{x\tau},\ B_{1\tau},\ B_{2\tau}\}$, where τ is treated as a parameter, and we construct the conjugate system $\{A_{x\tau}^*,\ B_{1\tau}^*,\ B_{2\tau}^*\}$. Assuming that the coefficients $a_1(x,\ \tau)$, $a_2(x,\ \tau)$, $a_3(x,\ \tau)$ are sufficiently smooth in x, we write the following differential expression, which is referred to as being formally conjugate to $A_{x\tau}T$:

$$A_{x\tau}^*\psi(x\ \ \tau)=\sum_{i=0}^{2}(-1)^i\frac{\partial^i(a_{3-i}\psi)}{\partial x^i}.$$

Let the functions $T(x,\ \tau)$, $\psi(x,\ \tau)$ have continuous derivatives with respect to x up to the second order inclusive in [0, b]. The second Green's relation is

$$\int_0^b(\psi A_{x\tau}T-TA_{x\tau}^*\psi)\,dx=[a_1\psi T_x+T(a_2\psi-(a_1\psi)_x)]_{x=0}^{x=b}. \tag{7}$$

The system of operators $\{A_{x\tau}^*,\ B_{1\tau}^*,\ B_{2\tau}^*\}$ is called conjugate to $\{A_{x\tau},\ B_{1\tau},\ B_{2\tau}\}$ if the right-hand side of (7) vanishes when

$$B_{1\tau}T(0,\ \tau)=0,\ B_{2\tau}T(b,\ \tau)=0; \tag{8}$$

$$B_{1\tau}^*\psi(0,\ \tau)=0,\ B_{2\tau}^*\psi(b,\ \tau)=0. \tag{9}$$

From (5), (6), and (8) we have

$$T_x(0,\ \tau)=-\frac{\sigma_1}{\gamma_1}T(0,\ \tau),\ T_x(b,\ \tau)=-\frac{\sigma_2}{\gamma_2}T(b,\ \tau).$$

We substitute these quantities into the right-hand side of (7) and equate the result to zero:

$$\left[-\frac{\sigma_2}{\gamma_2}a_1\psi T-T(a_1\psi)_x+a_2\psi T\right]_{x=b}+\left[\frac{\sigma_1}{\gamma_1}a_1\psi T-T(a_1\psi)_x+a_2\psi T\right]_{x=0}=0.$$

It then follows that

$$B_{1\tau}^*\psi(0,\ \tau)=\left[(a_1\psi)_x+\psi\left(a_1\frac{\sigma_1}{\gamma_1}-a_2\right)\right]_{x=0}, \tag{10}$$

$$B_{2\tau}^*\psi(b,\ \tau)=\left[(a_1\psi)_x+\psi\left(a_1\frac{\sigma_2}{\gamma_2}-a_2\right)\right]_{x=b}. \tag{11}$$

We determine now the operator $L_{x\tau}^*$, formally conjugate to $L_{x\tau}$:

$$L_{x\tau}^*=-\frac{\partial}{\partial\tau}-A_{x\tau}^* \tag{12}$$

and we consider the integral

$$\int_0^{\tau_m} d\tau \int_0^b (\psi L_{x\tau} T - T L^*_{x\tau}\psi)\, dx = -\int_0^{\tau_m} [a_1\psi T_x + T(a_2\psi - (a_1\psi)_x)]_{x=0}^{x=b} + \int_0^b [\psi T]_{\tau=0}^{\tau=\tau_m} dx. \quad (13)$$

The right-hand side of this expression is zero, if we assume the conditions (8) and (9) and also the conditions

$$T(x,\ 0) = 0,\ \psi(x,\ \tau_m) = 0.$$

Hence the following boundary-value problem is conjugate to the problem defined by (2)-(4):

$$L^*_{x\tau}\psi(x,\ \tau) = S(x,\ \tau),\ (x,\ \tau) \in Q; \quad (14)$$

$$\psi(x,\ \tau_m) = \zeta(x); \quad (15)$$

$$B^*_{1\tau}\psi(0,\ \tau) = g_1(\tau);\ B^*_{2\tau}\psi(b,\ \tau) = g_2(\tau), \quad (16)$$

where $S(x, \tau)$, $\zeta(x)$, $g_1(\tau)$, $g_2(\tau)$ are certain functions.

We introduce in the usual way the Green's functions for the systems of operators $\{L_{x\tau}, B_{1\tau}, B_{2\tau}\}$ and $\{L_{x\tau}^*, B_{1\tau}^*, B_{2\tau}^*\}$, denoting them by $G(x, \tau; x', \tau')$ and $G^*(x, \tau; x', \tau')$, respectively. These functions satisfy the conditions

$$L_{x\tau}G(x,\ \tau;\ x',\ \tau') = 0,\ x \in (0,\ b),\ \tau > \tau'; \quad (17)$$

$$B_{1\tau}G(0,\ \tau;\ x',\ \tau') = B_{2\tau}G(b,\ \tau;\ x',\ \tau') = 0,\ x' \in (0,\ b),\ \tau > \tau'; \quad (18)$$

$$\lim_{\varepsilon\to 0} \int_0^b G(x,\ \tau' + \varepsilon;\ x',\ \tau') f(x)\, dx = f(x'),\ f(x) \in C[0,\ b],\ \varepsilon > 0; \quad (19)$$

$$L^*_{x\tau}G^*(x,\ \tau;\ x',\ \tau') = 0,\ x \in (0,\ b),\ \tau < \tau'; \quad (20)$$

$$B^*_{1\tau}G^*(0,\ \tau;\ x',\ \tau') = B^*_{2\tau}G^*(b,\ \tau;\ x',\ \tau') = 0,\ x' \in (0,\ b),\ \tau < \tau'; \quad (21)$$

$$\lim_{\varepsilon\to 0} \int_0^b G^*(x,\ \tau' - \varepsilon;\ x',\ \tau') f(x)\, dx = f(x'),\ f(x) \in C[0,\ b],\ \varepsilon > 0. \quad (22)$$

It is known [13, 14] that for any two points (x, τ) and (x', τ') of Q for which $\tau > \tau'$,

$$G(x,\ \tau;\ x',\ \tau') = G^*(x',\ \tau';\ x,\ \tau). \quad (23)$$

The relation (23) can be obtained by putting

$$T(y,\ \tau) = G(y,\ t;\ x',\ \tau'),\ \psi(y,\ t) = G^*(y,\ t;\ x,\ \tau),$$

and integrating Green's identity with these values of T and ψ over the region

$$y \in [0+\delta,\ b-\delta],\ t \in [\tau'+\varepsilon,\ \tau-\varepsilon],\ \int_{\tau'+\varepsilon}^{\tau-\varepsilon} dt \int_{0+\delta}^{b-\delta} [G^*(y,\ t;\ x,\ \tau) L_{yt}G(y,\ t;\ x',\ \tau') - G(y,\ t;\ x',\ \tau') L^*_{yt}G^*(y, t;\ x,\ \tau)]\, dy = -\int_{\tau'+\varepsilon}^{\tau-\varepsilon} \{a_1(y,\ t)\, G^*(y,\ t;\ x,\ \tau) G_y(y,\ t;\ x',\ \tau') + G(y,\ t;\ x',\ \tau')\, [a_2(y,\ t)\, G^*(y,\ t;\ x,\ \tau) - (a_1(y,\ t)\, G^*(y,\ t;\ x,\ \tau))_y]\}|_{y=0+\delta}^{y=b-\delta} dt + \int_{0+\delta}^{b-\delta} [G(y,\ t;\ x',\ \tau') G^*(y,\ t;\ x,\ \tau)]_{t=\tau'+\varepsilon}^{t=\tau-\varepsilon}, \quad (24)$$

where ε, $\delta > 0$.

In this expression the integral on the left-hand side is equal to zero in view of the conditions (17) and (20). From (18), (5), (6) and (21), (10), (11), the first integral on the right-hand side of (24) will go to zero in the limit $\delta \to 0$. Therefore, we arrive at the relation

$$\int_0^b G(y,\ \tau-\varepsilon;\ x',\ \tau') G^*(y,\ \tau-\varepsilon;\ x,\ \tau)\, dy = \int_0^b G(y,\ \tau'+\varepsilon;\ x',\ \tau') G^*(y,\ \tau'+\varepsilon;\ x,\ \tau)\, dy,$$

from which we obtain the required relation (23) in the limit $\varepsilon \to 0$ if we use (19) and (22).

We consider again the integration of Green's identity, taking for the integration variables $x' \in [0, b]$, $\tau' \in [0, \tau - \varepsilon]$ and using the function $G^*(x', \tau'; x, \tau)$ as $\psi(x', \tau')$:

$$\int_0^{\tau-\varepsilon} d\tau' \int_0^b [G^*(x', \tau'; x, \tau) L_{x'\tau'} T(x', \tau') - T(x', \tau') L^*_{x'\tau'} G^*(x', \tau'; x, \tau)] dx' =$$

$$= -\int_0^{\tau-\varepsilon} \{a_1(x', \tau') G^*(x', \tau'; x, \tau) T_{x'}(x', \tau') - T(x', \tau')(a_1(x', \tau') G^*(x', \tau'; x, \tau))_{x'} + a_2(x', \tau') T(x', \tau') G^*(x', \tau'; x, \tau)\}|_{x'=0}^{x'=b} d\tau' -$$

$$- \int_0^b T(x', 0) G^*(x', 0; x, \tau) dx' + \int_0^b T(x', \tau - \varepsilon) G^*(x', \tau - \varepsilon; x, \tau) dx'.$$

Using (20) and (23), we obtain in the limit $\varepsilon \to 0$ the following expression:

$$T(x, \tau) = I_1(x, \tau) + I_2(x, \tau) + I_3(x, \tau), \tag{25}$$

where

$$I_1(x, \tau) = \int_0^\tau d\tau' \int_0^b G(x, \tau; x', \tau') L_{x'\tau'} T(x', \tau') dx' = \int_0^\tau d\tau' \int_0^b G(x, \tau; x', \tau') q(x', \tau') dx',$$

$$I_2(x, \tau) = \int_0^b T(x', 0) G(x, \tau; x', 0) dx' = \int_0^b G(x, \tau; x', 0) \xi(x') dx',$$

$$I_3(x, \tau) = \int_0^\tau \{a_1(x', \tau') T_{x'}(x', \tau') G^*(x', \tau'; x, \tau) - T(x', \tau') \times (a_1(x', \tau') G^*(x', \tau'; x, \tau))_{x'} + a_2(x', \tau') T(x', \tau') G^*(x', \tau'; x, \tau)\}|_{x'=0}^{x'=b} d\tau'.$$

In the formula for $I_3(x, \tau)$ we substitute the values of the derivatives $T_x(0, \tau')$ and $T_x(b, \tau')$ from the boundary conditions (4):

$$I_3(x, \tau) = \int_0^\tau a_1(b, \tau') G^*(b, \tau'; x, \tau) \frac{p_2(\tau')}{\gamma_2(\tau')} d\tau' - \int_0^\tau a_1(0, \tau') G^*(0, \tau'; x, \tau) \frac{p_1(\tau')}{\gamma_1(\tau')} d\tau' + \int_0^\tau \Big\{ T(0, \tau') \Big[a_1(0, \tau') \times G^*(0, \tau'; x, \tau) \frac{\sigma_1(\tau')}{\gamma_1(\tau')} + (a_1(0, \tau') G^*(0, \tau'; x, \tau))_{x'} - a_2(0, \tau') G^*(0, \tau'; x, \tau) \Big] - T(b, \tau') \Big[a_1(b, \tau') G^*(b, \tau'; x, \tau) \frac{\sigma_2(\tau')}{\gamma_2(\tau')} + (a_1(b, \tau') G^*(b, \tau'; x, \tau))_{x'} - a_2(b, \tau') G^*(b, \tau'; x, \tau) \Big] \Big\} d\tau'.$$

Using the conditions (23) and (21), (10), (11), we obtain finally

$$I_3(x, \tau) = \int_0^\tau G(x, \tau; b, \tau') a_1(b, \tau') \frac{p_2(\tau')}{\gamma_2(\tau')} d\tau' - \int_0^\tau G(x, \tau; 0, \tau') a_1(0, \tau') \frac{p_1(\tau')}{\gamma_1(\tau')} d\tau'.$$

Therefore, we have found expressions for the three terms I_1, I_2, and I_3 which determine the solution $T(x, \tau)$ (from the superposition result (25)) of the boundary-value problem (2)-(4) in terms of the Green's function.

If instead of the boundary conditions used above we were given conditions of the first kind ($\gamma_1 = \gamma_2 = 0$, $\sigma_1 = \sigma_2 = 1$), then the solution (25) would change only in the term I_3, which in this case would take the form

$$I_3(x, \tau) = \int_0^\tau p_1(\tau') (a_1(0, \tau') G^{\mathrm{I}}(x, \tau; 0, \tau'))_{x'} d\tau' - \int_0^\tau p_2(\tau') (a_1(b, \tau') G^{\mathrm{I}}(x, \tau; b, \tau'))_{x'} d\tau',$$

where G^{I} is the Green's function for (2) with boundary conditions of the first kind.

If we now consider any of the functions $q(x, t)$, $\xi(x)$, $p_1(\tau)$, $p_2(\tau)$ as unknown, we arrive at a formulation of the inverse problem with a linear, given operator A in the functional (1). In this case an explicit expression can be obtained for the error gradient $J' = A^*(Au - f)$, defining the conjugate operator A^*.

Let the required function be $q(x, \tau)$. From the point of view of the uniqueness of the solution of this inverse problem, it is necessary to use the field $T(x, \tau)$ as the initial data, and write the error in the form

$$J(q) = \frac{1}{2}\int_0^{\tau_m} d\tau \int_0^b (Aq - f)^2 dx,$$

where

$$Aq = \int_0^{\tau} d\tau' \int_0^b q(x', \tau') G(x, \tau; x', \tau')\, dx', \quad A: L_2(Q) \to L_2(Q);$$

$$f = T(x, \tau) - I_2(x, \tau) - I_3(x, \tau).$$

We introduce the notation $\Delta(x, \tau) = Aq - f$, and using the definition of the conjugate operator, we write the following identity for the scalar products:

$$(Aq, \Delta)_{L_2(Q)} = (q, A^*\Delta)_{L_2(Q)}. \tag{26}$$

We write out the left-hand side of (26):

$$(Aq, \Delta)_{L_2(Q)} = \int_0^{\tau_m} d\tau \int_0^b \Delta(x, \tau)\, dx \int_0^{\tau} d\tau' \int_0^b q(x', \tau') G(x, \tau; x', \tau')\, dx'$$

and change the order of integration

$$(Aq, \Delta)_{L_2(Q)} = \int_0^{\tau_m} d\tau' \int_0^b q(x', \tau')\, dx' \int_{\tau'}^{\tau_m} d\tau \int_0^b \Delta(x, \tau) G(x, \tau; x', \tau')\, dx. \tag{27}$$

Comparing (27) with (26), we obtain the required expression for the gradient

$$J'_{q(x', \tau')} = A^*\Delta = \int_{\tau'}^{\tau_m} d\tau \int_0^b \Delta(x, \tau) G(x, \tau; x', \tau')\, dx.$$

We assume now that $q(x, \tau) = \varphi(x, \tau)s(\tau)$, where $\varphi(x, \tau)$ is known, and assume that we are given the functions $\tilde{f}_n(\tau) = T(d_n, \tau)$, $n = \overline{1, N}$, $N \geq 1$, $0 \leq d_n \leq b$. It is required to find the gradient $J'_{s(\tau)}$ of the functional

$$J(s) = \frac{1}{2}\sum_{n=1}^{N} \int_0^{\tau_m} (A_{sn}s - f_n)^2\, d\tau,$$

where

$$A_{sn}s = \int_0^{\tau} d\tau' \int_0^b \varphi(x', \tau') s(\tau') G(d_n, \tau; x', \tau')\, dx';$$

$$A_{sn}: L_2[0, \tau_m] \to L_2[0, \tau_m]; \quad f_n = \tilde{f}_n(\tau) - I_2(d_n, \tau) - I_3(d_n, \tau).$$

It is not difficult to show that $J'_s = \sum_{n=1}^{N} A^*_{sn}(A_{sn}s - f_n)$, therefore it is necessary to find the conjugate operators A^*_{sn}. Writing the identity

$$(A_{sn}s, \Delta_{sn})_{L_2[0, \tau_m]} = (s, A^*_{sn}\Delta_{sn})_{L_2[0, \tau_m]}, \tag{28}$$

where $\Delta_{sn}(\tau) = A_{sn}s - f_n$, and writing out the left-hand side of (28), we obtain the following result after changing the order of integration

$$(A_{sn}s, \Delta_{sn}) = \int_0^{\tau_m} s(\tau')\, d\tau' \int_{\tau'}^{\tau_m} \Delta_{sn}(\tau) \Phi(d_n, \tau, \tau')\, d\tau,$$

where

$$\Phi(d_n, \tau, \tau') = \int_0^b \varphi(x', \tau') G(d_n, \tau; x', \tau')\, dx'.$$

Hence the required result for the conjugate operator is

$$A^*_{sn}\Delta_{sn} = \int_{\tau'}^{\tau_m} \Delta_{sn}(\tau)\Phi(d_n, \tau, \tau')\, d\tau.$$

Following the same reasoning, it can be shown that if $q(x, \tau) = \varphi(x, \tau)w(x)$ ($\varphi(x, \tau)$ is a known function) then the gradient of the functional

$$J(w) = \frac{1}{2}\sum_{n=1}^{N}\int_0^{\tau_m} (A_{wn}w - f_n)^2 d\tau,$$

where

$$A_{wn}w = \int_0^{\tau} d\tau' \int_0^b \varphi(x', \tau') w(x') G(d_n, \tau; x', \tau')\, dx', \quad A_{wn}: L_2[0, b] \to L_2[0, \tau_m],$$

will be equal to

$$J'_w = \sum_{n=1}^{N} A^*_{wn}\Delta_{wn}, \quad A^*_{wn}\Delta_{wn} = \int_0^{\tau_m} \Delta_{wn} K(d_n, x', \tau)\, d\tau,$$

$$\Delta_{wn} = A_{wn}w - f_n, \quad K(d_n, x', \tau) = \int_0^{\tau} \varphi(x', \tau') G(d_n, \tau; x', \tau')\, d\tau'.$$

Similarly we can derive expressions for the gradient of the error functionals with respect to $\xi(x)$, $p_1(\tau)$, $p_2(\tau)$. Omitting the subscript n for simplicity, we write down the corresponding formulas for the operators A and A*:

$$A_\xi \xi = \int_0^b \xi(x') G(d, \tau; x', 0)\, dx', \quad A_\xi : L_2[0, b] \to L_2[0, \tau_m];$$

$$A^*_\xi y = \int_0^{\tau_m} y(\tau) G(d, \tau; x', 0)\, d\tau;$$

$$A_{p_1} p_1 = -\int_0^{\tau} a_1(0, \tau') \frac{p_1(\tau')}{\gamma_1(\tau')} G(d, \tau; 0, \tau')\, d\tau', \quad A_{p_1} : L_2[0, \tau_m] \to L_2[0, \tau_m];$$

$$A^*_{p_1} y = -\frac{a_1(0, \tau')}{\gamma_1(\tau')} \int_{\tau'}^{\tau_m} y(\tau) G(d, \tau; 0, \tau')\, d\tau;$$

$$A_{p_2} p_2 = \int_0^{\tau} a_1(b, \tau') \frac{p_2(\tau')}{\gamma_2(\tau')} G(d, \tau; b, \tau')\, d\tau', \quad A_{p_2} : L_2[0, \tau_m] \to L_2[0, \tau_m];$$

$$A^*_{p_2} y = \frac{a_1(b, \tau')}{\gamma_2(\gamma')} \int_{\tau'}^{\tau_m} y(\tau) G(d, \tau; b, \tau')\, d\tau.$$

Similarly for boundary conditions of the first kind, we obtain

$$A^*_{p_1} y = \int_{\tau'}^{\tau_m} y(\tau) (a_1(0, \tau') G^{\mathrm{I}}(d, \tau; 0, \tau'))_{x'} d\tau;$$

$$A^*_{p_2} y = -\int_{\tau'}^{\tau_m} y(\tau) (a_1(b, \tau') G^{\mathrm{I}}(d, \tau; b, \tau'))_{x'} d\tau. \tag{29}$$

It was assumed above that (2) was specified with fixed boundaries and it was also assumed that the functions $f_n(\tau)$ were known at certain points d_n, fixed in position. In practice it is sometimes necessary to take into account the movement of both the boundaries of the region and also the measured points. That is, we require the solution of the inverse problem in the region $\underline{Q_\tau} = \{X_1(\tau) \leq x \leq X_2(\tau),\ 0 \leq \tau \leq \tau_m\}$, knowing the functions $\bar{f}_n(\tau) = T(d_n(\tau), \tau)$, $n = \overline{1, N}$.

One of the possible approaches to the solution of this problem is based on the method of fictitious boundaries [3]. The essence of the method, in the context of the case considered here, is as follows. The region Q_τ is expanded to the rectangular region $\bar{Q} = [a, b] \times [0, \tau_m]$, where $a = \min_{\tau \in [0, \tau_m]} X_1(\tau)$, $b = \max_{\tau \in [0, \tau_m]} X_2(\tau)$ are so-called fictitious boundaries. It is then required to determine the coefficients of equation (2) in such a way that the required smoothness of these coefficients as functions of x, τ will be assured over the entire region $\bar{Q}$. From the solution of the auxiliary inverse boundary-value problem in the region $\bar{Q}$ with the data $\bar{f}_n(\tau)$, $n = \overline{1, N}$, the conditions on the boundaries a and b are found. We note that it is convenient to consider boundary conditions of the first kind, i.e., to look for the functions $T(a, \tau)$, $T(b, \tau)$, since a problem of this kind for (2) is better determined than the other formulations [3]. Then from the solution of the boundary-value problem of the first kind in the region Q, the required quantities are determined along the lines $X_1(\tau)$ and $X_2(\tau)$, $\tau \in [0, \tau_m]$.

The transition to an expanded rectangular region with fictitious boundaries is useful in simplifying the solution algorithm of the original inverse problem, since for many cases (Eq. (2) with constant coefficients, for example) the Green's function can be obtained in explicit form (see [15], for example). In the formulas for the error gradient, $d_n(\tau)$ must be substituted for the fixed constants d_n.

It must be pointed out that the expansion of the region of the solution decreases the determinability of the inverse problem to be solved in this region. Therefore it is best to choose an expanded region as close as possible to the original region and yet such that the Green's function is known for it.

We consider one of the practically important cases: the inverse problem for the heat equation with constant coefficients in a region with one moving and one fixed boundary:

$$T_\tau = aT_{xx},\ (x, \tau) \in Q_\tau = \{0 < x < X(\tau),\ 0 < \tau \leqslant \tau_m\}.$$

We take the boundary conditions in the form

$$T(x, 0) = \xi(x),\ x \in [0,\ X(0)];$$

$$T(0, \tau) = f(\tau),\ \gamma T_x(X(\tau), \tau) + \sigma T(X(\tau), \tau) = p(\tau),\ \tau \in [0, \tau_m].$$

We take as an expanded region the following trapezoidal region $Q = \{0 < x < y(\tau) = \ell + v\tau,\ 0 < \tau \leq \tau_m\}$ closest to Q_τ. We thus require that the function $y(\tau)$ satisfy the conditions

$$y(\tau) \geqslant X(\tau),\ y(\tau): \min_{l, v} \max_{\tau} |y(\tau) - X(\tau)|,$$

where $\ell \geq 0$ and v are parameters determining the uniformly moving fictitious boundary.

We consider the boundary-value problem of the first kind for the heat equation in the region Q, with the condition $T(y(\tau), \tau) = \varkappa(\tau)$ on the fictitious boundary. Following [16], we write the solution

$$T(x, \tau) = \int_0^{y(0)} \xi(x') G^{\mathrm{I}}(x, \tau;\ x', 0)\, dx' + a \int_0^{\tau} f(\tau') G^{\mathrm{I}}_{x'}(x, \tau;\ 0, \tau')\, d\tau' - a \int_0^{\tau} \varkappa(\tau') G^{\mathrm{I}}_{x'}(x, \tau;\ y(\tau'), \tau')\, d\tau'. \quad (30)$$

In this expression the Green's function has the form

$$G^{\mathrm{I}}(x, \tau;\ x', \tau') = \frac{1}{2\sqrt{a\pi(\tau - \tau')}} \sum_{j=-\infty}^{j=+\infty} \exp\left[-\frac{v^2\tau'}{a} j^2 - \frac{x'v}{a} j\right] \times$$

$$\times \left\{\exp\left[-\frac{(2v\tau' j + x' - x)^2}{4a(\tau - \tau')}\right] - \exp\left[-\frac{(2v\tau' j + x' + x)^2}{4a(\tau - \tau')}\right]\right\}.$$

Hence in the calculation of the gradient of the functional $J'\varkappa(\tau)$ in the iterative solution of the inverse problem one can use (30) for the function $\varkappa(\tau)$ and (29) for the conjugate operator, in which we put in place of $a_1(b, \tau')$ and b, the constant a and the function $y(\tau')$, respectively. If the temperature is measured at the moving points $d_n(\tau)$, then the number d is replaced by the corresponding function.

After calculating $\varkappa(\tau)$, the boundary-value problem of the first kind is solved in the region Q, and the required quantity $p(\tau)$ is found on the line $X(\tau)$. The integral representation (30) is also used for this procedure.

We note that Green's functions have been obtained for the heat equation with constant coefficients for other regions with uniformly moving boundaries [16].

Therefore, the method discussed in this section is convenient to use in the calculation of the error gradient whenever the Green's function can be found analytically. In the case when the Green's function is unknown and can be constructed only numerically, it is more efficient to find the gradient by a different, more general method which uses the solution of the conjugate boundary-value problem. This problem will be considered in the next paper of the current series.

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